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A PERTURBATION ANALYSIS OF MODE INTERACTION IN POSTBUCKLING BEHAVIOR AND IMPERFECTION SENSITIVITY

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Abstract—A perturbation method is presented for the analysis of postbuckling behavior and imperfection sensitivity of elastic structures which have more than one buckling mode. The method is exemplified by analyzing a complete spherical shell under external hydrostatic pressure. For this analysis, use is made of two- and three-mode models of the shell derived earlier from the shallow shell equations.

1. INTRODUCTION

In recent years there has been some interest in the analysis of conservative elastic systems whose loss of stability is associated with the existence of more than one buckling mode at the lowest critical point on the fundamental path. Not only can the multiplicity of buckling modes arise naturally in a problem, the coincidence of the two lowest buckling loads may sometimes be used as the optimization criterion in structural design. Traditionally, the general procedure for analyzing the imperfection sensitivity of such structures consists of two steps, as, for example, in Hutchinson[1]. Firstly, a reduced set of equilibrium equations in terms of the amplitude of the buckling modes is obtained by using the Koiter's method[2]. The equilibrium path of the structure is then calculated by solving these nonlinear equations. Once the equilibrium path is known it is a simple matter to calculate a limit point or a bifurcation point.

In the following presentation a somewhat different approach is used. Its distinguishing feature is that the equations governing the equilibrium and the transition to instability are solved simultaneously to obtain the critical load of the imperfect structure. Since it is usually sufficient to calculate only the instability load for a given imperfection and the knowledge of the equilibrium path of the imperfect structure is not, in itself, very important, the present method has an obvious advantage.

The notation used in the development is the one introduced by Koiter[2] because of its compactness and applicability to both continuous and discrete systems. Even the reduced set of equilibrium equations derived by Koiter's method can be effectively treated by using the approach outlined here. In fact, the examples that have been chosen to illustrate the method—two and three-mode models of externally pressurized spherical shells—are the results of applying Koiter's method to the continuum problem.

For the sake of completeness a perturbation method to calculate the equilibrium paths of the perfect structure is also presented. This formulation is slightly different from the ones currently used, e.g.[3].

In the sequel it is constantly required to use the solvability conditions for a singular set of equations. Let the set of equations

$$P_{11}(w, \delta w) = 0 \tag{1.1}$$

have r nontrivial solutions ϕ_i, i.e.

$$P_{11}(\phi_i, \delta w) = 0, \quad i = 1, \dots, r. \tag{1.2}$$

Then the nonhomogeneous set of equations corresponding to (1.1),

$$P_{11}(z, \delta w) = R(\delta w) \tag{1.3}$$

has a solution if and only if

$$R(\phi_i) = 0, \quad i = 1, \ldots, r. \tag{1.4}$$

Once the condition (1.4) is satisfied eqn (1.3) can be solved either by using a generalized inverse of a singular matrix in case of discrete systems or by the use of generalized Green's matrix for continuum problems.

2. PERTURBATION METHOD FOR MULTIPLE BIFURCATION-POSTBUCKLING ANALYSIS

The equilibrium equations are assumed in the form

$$\delta P(u) = P_{11}(u, \delta u) + (\lambda - \lambda_0)P'_{11}(u, \delta u) + (\lambda - \lambda_0)^2 P''_{11}(u, \delta u) \cdots + P_{21}(u, \delta u) + (\lambda - \lambda_0)P'_{21}(u, \delta u) + \cdots + P_{31}(u, \delta u) + \cdots = 0. \tag{2.1}$$

It is assumed that there is a bifurcation point at load level λ_0 with r bifurcation modes, orthonormalized with respect to a positive definite quadratic form, that is, with δ_{ij} denoting the Kronecker delta,

$$T_{11}(\varphi_i, \varphi_j) = \delta_{ij}, \quad i,j = 1, r. \tag{2.2}$$

The bifurcating branches from load level λ_0 are expanded in the form

$$u = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 \ldots, \tag{2.3a}$$

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 \ldots, \tag{2.3b}$$

with

$$v_1 = \sum_{j=1}^{r} \alpha_j\varphi_j, \tag{2.4a}$$

$$\sum_{j=1}^{r} \alpha_j^2 = 1, \tag{2.4b}$$

$$T_{11}(v_1, v_j) = \delta_{1j}. \tag{2.4c}$$

On substituting the expansion (2.3) into eqn (2.1) and equating the same powers of ϵ a sequence of linear problems is obtained:

$$O(\epsilon) \quad P_{11}\left(\sum_{j=1}^{r} \alpha_j\varphi_j, \delta u\right) = 0, \tag{2.5a}$$

$$O(\epsilon^2) \quad P_{11}(v_2, \delta u) + \lambda_1 P'_{11}\left(\sum_{j=1}^{r} \alpha_j\varphi_j, \delta u\right) + P_{21}\left(\sum_{j=1}^{r} \alpha_j\varphi_j, \delta u\right) = 0, \tag{2.5b}$$

$$O(\epsilon^3) \quad P_{11}(v_3, \delta u) + \lambda_1[P'_{11}(v_2, \delta u) + \lambda_1 P''_{11}(v_1, \delta u)] + \lambda_2 P'_{11}(v_1, \delta u) + P_{111}(v_1, v_2, \delta u) + \lambda_1 P'_{21}(v_1, \delta u) + P_{31}(v_1, \delta u) = 0. \tag{2.5c}$$

The first of these is satisfied since φ_j's are the bifurcation modes. The condition that eqn (2.5b) be solvable for v_2 leads to a set of nonlinear algebraic equations

$$\lambda_1 P'_{11}\left(\sum_{j=1}^{r} \alpha_j\varphi_j, \varphi_i\right) + P_{21}\left(\sum_{j=1}^{r} \alpha_j\varphi_j, \varphi_i\right) = 0. \tag{2.6}$$

Defining the matrices

$$A_{ij} = P'_{11}(\varphi_i, \varphi_j), \tag{2.7}$$

$$2[M1(\alpha)]_{ij} = \sum_{k=1}^{r} \alpha_k P_{111}(\varphi_i, \varphi_k, \varphi_j), \tag{2.8}$$

the eqn (2.6) can be written as

$$\lambda_1 A\alpha + M1(\alpha)\alpha = 0, \tag{2.9}$$

with the constraint (2.4b)

$$\alpha^T\alpha = 1 \tag{2.10}$$

in terms of the vector α^T with components $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. It is assumed that the matrix A is nonsingular. In that case and if $M1(\alpha)$ is not identically zero, the set (4.9)–(4.10) will have s solutions denoted by

$$\alpha_b{}^I, \lambda_{1_b}^I; \quad I = 1, \ldots, s. \tag{2.11}$$

For each of these branches solution of eqn (2.5b) is of the form

$$v_2{}^I = \bar{v}_2{}^I + \sum_{j=1}^{r} \beta_j{}^I \varphi_j, \tag{2.12}$$

where $\bar{v}_2{}^I$ is the unique solution of

$$P_{11}(\bar{v}_2{}^I, \delta u) + \lambda_{1_b}^I P'_{11}\left(\sum_{j=1}^{r} \alpha_j{}^I\varphi_j, \delta u\right) + P_{21}\left(\sum_{j=1}^{r} \alpha_j{}^I\varphi_j, \delta u\right) \tag{2.13a}$$

$$T_{11}(\bar{v}_2{}^I, \varphi_j) = 0; \quad j = 1, \ldots, r; \quad I = 1, \ldots, s. \tag{2.13b}$$

Using the condition (2.4c) with $j = 2$, one also gets

$$\sum_{j=1}^{r} \beta_j{}^I\alpha_j{}^I = 0. \tag{2.14}$$

Equation (2.12) is now substituted into the condition that eqn (2.5c) be solvable for v_3 for each of the branches. With the definition

$$\begin{aligned}\gamma_j{}^I \equiv {} & (\lambda_{1_b}^I)^2 P''_{11}(v_1{}^I, \varphi_j) + \lambda_1{}^I P'_{21}(v_1{}^I, \varphi_j) + P_{31}(v_1{}^I, \varphi_j) \\ & + \lambda_1{}^I P'_{11}(\bar{v}_2{}^I, \varphi_j) + P_{111}(v_1{}^I, \bar{v}_2, \varphi_j),\end{aligned} \tag{2.15a}$$

$$\bar{\gamma}_j{}^I \equiv P'_{11}(v_1{}^I, \varphi_j), \tag{2.15b}$$

this solvability condition becomes

$$[\lambda_{1_b}^I A + 2M1(\alpha_b{}^I)]\beta^I + \lambda_{2_b}^I\bar{\gamma}^I + \gamma^I = 0. \tag{2.16}$$

Equations (2.16) and (2.14) are $(r+1)$ linear equations for the r dimensional vector β^I and scalar $\lambda_2{}^I$, thus

$$\begin{bmatrix} \lambda_{1_b}^I A + 2M1(\alpha_b{}^I) & \bar{\gamma}^I \\ (\alpha_b{}^I)^T & 0 \end{bmatrix} \begin{bmatrix} \beta^I \\ \lambda_{2_b}^I \end{bmatrix} + \begin{bmatrix} \gamma^I \\ 0 \end{bmatrix} = 0. \tag{2.17}$$

Similar linear equations are obtained in calculating higher order terms; thus it is required that $(r+1)\times(r+1)$ matrix in eqn (2.17) be nonsingular.

This completes the basic procedure for the analysis of postbuckling behavior which could, in principle, be carried to any order. If $M1(\alpha) = 0$, eqn (2.9) yields $\lambda_1 = 0$ and the vector α

undetermined. However the third order equation yields a cubic set of equations of the form

$$\lambda_2 A\alpha + M2(\alpha)\alpha = 0, \tag{2.18a}$$

$$\alpha^T\alpha = 1. \tag{2.18b}$$

Thus the analysis essentially involves solving a set of nonlinear algebraic equations for obtaining the lowest order term in the perturbation expansion. This establishes the number of branches passing through the bifurcation point. One then proceeds along each of these branches and only linear analysis is required for obtaining higher order terms.

3. PERTURBATION METHOD FOR MULTIPLE BIFURCATION—IMPERFECTION SENSITIVITY

The system described by the equilibrium eqns (2.1) is now assumed to have an imperfection. The equilibrium eqns (2.1) are thus modified to

$$\begin{aligned}\delta P(u) = {} & P_{11}(u, \delta u) + (\lambda - \lambda_0)P'_{11}(u, \delta u) + (\lambda - \lambda_0)^2 P''_{11}(u, \delta u) + P_{21}(u, \delta u) + (\lambda - \lambda_0)P'_{21}(u, \delta u) \\ & + P_{31}(u, \delta u) + \mu[Q_1(\delta u) + (\lambda - \lambda_0)Q'_1(\delta u) + Q_{11}(u, \delta u)] = 0. \end{aligned} \tag{3.1}$$

Rather than try to solve eqn (3.1) for each μ and then obtain the instability loads—which is the essence of Koiter's method—attention is restricted in the following to solve eqn (3.1) simultaneously with the following eqn (3.2), which is the necessary condition that the load level λ correspond to a critical point on the equilibrium path. Thus one has

$$\begin{aligned}\delta^2 P(u) = {} & P_{11}(\psi, \delta u) + (\lambda - \lambda_0)P'_{11}(\psi, \delta u) + (\lambda - \lambda_0)^2 P''_{11}(\psi, \delta u) + P_{111}(u, \psi, \delta u) \\ & + (\lambda - \lambda_0)P'_{111}(u, \psi, \delta u) + P_{211}(u, \psi, \delta u) + \mu[Q_{11}(\psi, \delta u)] = 0, \end{aligned} \tag{3.2a}$$

$$T_{11}(\psi, \psi) \neq 0. \tag{3.2b}$$

Solution of eqns (3.1), (3.2) is expanded in the form (see Appendix)

$$\begin{aligned} u &= \epsilon v_1 + \epsilon^2 v_2 + \cdots, \\ \psi &= \psi_0 + \epsilon\psi_1 + \cdots, \\ \lambda &= \lambda_0 + \epsilon\lambda_1 + \cdots, \\ \mu &= \epsilon^2\mu_2 + \cdots, \\ T_{11}(v_1, v_{j_1}) &= T_{11}(\psi_0, \psi_{j_2}) = 0; \quad j_1 \geq 2, j_2 \geq 1, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} v_1 &= \sum_{i=1}^{r} \alpha_i\varphi_i, \\ \psi_0 &= \sum_{i=1}^{r} \xi_i\varphi_i, \\ \sum_{i=1}^{r} \alpha_i^2 &= \sum_{i=1}^{r} \xi_i^2 = 1. \end{aligned} \tag{3.4}$$

Substitution of expansion (3.3) into the eqns (3.1), (3.2) leads to a sequence of linear problems. As in the last section, the first one is identically satisfied for both eqns (3.1) and (3.2). The second set is

$$P_{11}(v_2, \delta u) + \lambda_1 P'_{11}(v_1, \delta u) + P_{21}(v_1, \delta u) + \mu_2 Q_1(\delta u) = 0, \tag{3.5a}$$

$$P_{11}(\psi_1, \delta u) + \lambda_1 P'_{11}(\psi_0, \delta u) + P_{111}(v_1, \psi_0, \delta u) = 0. \tag{3.5b}$$

The condition that these must be solvable for v_2 and ψ_1 leads to a set of algebraic equations

$$\lambda_1 P'_{11}\left(\sum_{i=1}^{r} \alpha_i\varphi_i, \varphi_j\right) + P_{21}\left(\sum_{i=1}^{r} \alpha_i\varphi_i, \varphi_j\right) + \mu_2 Q_1(\varphi_j) = 0, \tag{3.6a}$$

$$\lambda_1 P'_{11}\left(\sum_{i=1}^{r} \xi_i\varphi_i, \varphi_j\right) + P_{111}\left(\sum_{i=1}^{r} \varphi_i\varphi_i, \sum_{i=1}^{r} \xi_i\varphi_i, \varphi_j\right) = 0, \tag{3.6b}$$

where the representation (3.4a), (3.4b) has also been used. Using the matrix notation of the last section, this set can be written in the convenient form

$$\lambda_1 A\alpha + M1(\alpha)\alpha + \mu_2\zeta = 0; \quad \zeta_j \equiv Q_1(\varphi_j), \tag{3.7a}$$

$$\lambda_1 A\xi + 2M1(\alpha)\xi = 0, \tag{3.7b}$$

and from eqn (3.4), vectors α and ξ, with components α_j and ξ_j respectively, also satisfy

$$\alpha^T\alpha = \xi^T\xi = 1. \tag{3.8a, b}$$

Being nonlinear, it is difficult to solve these equations directly for a given ζ. In the following a method for obtaining these solutions is presented.

Usually it is of importance to determine the worst case, i.e. for a given magnitude of the vector ζ, the combination of the various components ζ_j which produces the maximum deterioration in the buckling load is required. Without any loss of generality the magnitude can be taken to be unity. Thus defining a vector

$$\bar{\zeta} = \mu_2\zeta, \tag{3.9a}$$

so that

$$\mu_2 = \sqrt{[(\bar{\zeta}^T\bar{\zeta})]}, \tag{3.9b}$$

the eqn (3.7a) transforms to

$$\lambda_1 A\alpha + M1(\alpha)\alpha + \bar{\zeta} = 0. \tag{3.10}$$

The problems (3.7), (3.8) are now equivalent to eqns (3.7b), (3.8) and (3.10).

A class of solutions to this modified problem is at once obvious, thus (with subscript μ distinguishing the solution for the imperfect case from the perfect one):

$$\lambda_{1_\mu}^I = 2\lambda_{1_b}^I, \tag{3.11a}$$

$$\xi^I = \alpha_b{}^I, \tag{3.11b}$$

$$\alpha_\mu{}^I = \alpha_b{}^I, \tag{3.11c}$$

$$\bar{\zeta}^I = -\lambda_{1_b}^I A\alpha_b{}^I \text{ (no sum)}. \tag{3.11d}$$

That these are indeed the solutions can be verified by direct substitution and using the fact that the quantities $\lambda_{1_b}^I$ and $\alpha_b{}^I$ defined by eqn (2.11) satisfy the eqns (2.9).

Another way of generating the solutions is as follows: first a unit vector $\alpha = \bar{\alpha}$ is assumed. The vector ξ and λ_1 can obtained by solving the eigenvalue problem

$$Ax + \chi M1(\bar{\alpha})x = 0, \tag{3.12a}$$

$$x^T x = 1. \tag{3.12b}$$

Since the matrices A and $M1(\bar{\alpha})$ are real and symmetric, the eigenvalues are real. Hence corresponding to this $\bar{\alpha}$, from eqns (3.7b) and (3.10), the solution is

$$\bar{\lambda}_1 = 2/\chi, \tag{3.13a}$$

$$\bar{\xi} = x, \tag{3.13b}$$

$$\bar{\zeta} = -[\bar{\lambda}_1 A\bar{\alpha} + M1(\bar{\alpha})\bar{\alpha}]. \tag{3.13c}$$

Thus, in principle, a continuum of solutions can be generated and from these the imperfection that leads to the maximum decrease in the buckling load can be determined.

It must be emphasized here that the potential energy formulation has been used here only for notational convenience—this perturbation method can be applied to all self adjoint boundary value problems and with a slight modification (i.e. using the null vector of the adjoint problem in the solvability condition) to non-self adjoint problems also.

In case the matrix $M1(\alpha)$ is identically zero, this analysis yields $\lambda_1 = \mu_2 = 0$. Hence higher order expansion has to be used. This will yield equations of the form

$$\lambda_2 A\alpha + M2(\alpha)\alpha + \mu_3\zeta = 0, \tag{3.14a}$$

$$\lambda_2 A\xi + 3M2(\alpha)\xi = 0. \tag{3.14b}$$

Again, a procedure similar to the one described above can be used for solution of these equations.

For the subsequent set of linear equations for higher order terms to be solvable a condition analogous to eqn (2.17) has to be satisfied. Such a condition can be obtained by taking the derivative of the left hand side of eqns (3.7), (3.8) with respect to the quantities α, ξ, λ and μ_2 and requiring that the matrix so obtained be nonsingular. Thus the $(2r+2)\times(2r+2)$ matrix

$$\begin{bmatrix} \lambda_1 A + 2M1(\alpha) & 0 & A\alpha & \zeta \\ 2M1(\xi) & \lambda_1 A + 2M1(\alpha) & A\xi & 0 \\ 2\alpha^T & 0 & 0 & 0 \\ 0 & 2\xi^T & 0 & 0 \end{bmatrix}.$$

evaluated at the solution of eqns (3.7) and (3.8) must be nonsingular for the assumed perturbation expansion to be valid.

4. AN EXAMPLE OF DEGENERATE BIFURCATION—THE EXTERNALLY PRESSURIZED SPHERICAL SHELL

To illustrate the use of the perturbation method for case of degenerate bifurcation, the imperfection sensitivity of an externally pressurized shell has been analyzed. The continuum model—for example, the shallow shell equations used by Hutchinson[1]—is characterized by the existence of large number of modes associated with the bifurcation point. The problem becomes tractable if one imposes the requirement that only a finite number of modes participate in the initial postbuckling regime. This could be done in a number of seemingly different ways, as in Hutchinson[1] or Reissner[4]. For the sake of brevity it is not attempted here to proceed from the shallow shell equations and solution is obtained using some results of Hutchinson whose work can be referred to for details.

If a two-mode solution is assumed, the governing equations in terms of nondimensional variables are

$$u_1 - \lambda u_1 - Du_2^2 = \lambda\mu f_1, \tag{4.1a}$$

$$\frac{1}{2}(1-\lambda)u_2 - 2Du_1u_2 = \frac{1}{2}\lambda\mu f_2, \tag{4.1b}$$

where $f^T = (f_1, f_2)$ is a unit vector. The quantities λ and D are defined in terms of critical pressure p_c and Poissons ratio ν, thus,

$$\lambda \equiv p/p_c, \tag{4.2a}$$

$$D = \frac{9}{32}\sqrt{(3(1-\nu^2))}. \tag{4.2b}$$

The bifurcation point is at $\lambda_0 = 1$ with the corresponding bifurcation modes

$$\varphi_1{}^T = [1, 0]; \quad \varphi_2{}^T = [0, 1]. \tag{4.3a, b}$$

The necessary condition that solutions (4.1) be critical at load level λ is the existence of a solution of

$$\psi_1 - \lambda\psi_1 - 2D\psi_2 u_2 = 0, \tag{4.4a}$$

$$\frac{1}{2}(1-\lambda)\psi_2 - 2Du_2\psi_1 - 2Du_1\psi_2 = 0, \tag{4.4b}$$

$$\psi_1{}^2 + \psi_2{}^2 \neq 0. \tag{4.4c}$$

Solution of eqns (4.1) and (4.4) is assumed in the form

$$u = \binom{u_1}{u_2} = \epsilon(\alpha_1\varphi_1 + \alpha_2\varphi_2) + O(\epsilon^2), \tag{4.5a}$$

$$\lambda = 1 + \lambda_1\epsilon + O(\epsilon^2), \tag{4.5b}$$

$$\mu = \epsilon^2\mu_2 + O(\epsilon^3), \tag{4.5c}$$

$$\psi = \binom{\psi_1}{\psi_2} = (\xi_1\varphi_1 + \xi_2\varphi_2) + O(\epsilon), \tag{4.5d}$$

with

$$\alpha_1{}^2 + \alpha_2{}^2 = 1, \tag{4.6a}$$

$$\xi_1{}^2 + \xi_2{}^2 = 1, \tag{4.6b}$$

Substitution of the form (4.5) and (4.6) into the original equation leads to the following lowest order equations

$$-\lambda_1\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} + \begin{bmatrix}0 & -D\alpha_2\\ -D\alpha_2 & -D\alpha_1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \mu_2\begin{pmatrix}f_1\\ \frac{1}{2}f_2\end{pmatrix}, \tag{4.7a}$$

$$-\lambda_1\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + 2\begin{bmatrix}0 & -D\alpha_2\\ -D\alpha_2 & -D\alpha_1\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = 0. \tag{4.7b}$$

With the constraint (4.6), these equations have a one parameter family of solutions with α_1 in the interval $[-1, 1]$. Of interest is the so-called imperfection sensitivity coefficient $(\lambda_1/\sqrt{(\mu_2)})$, since an elimination of ϵ in eqns (4.5b, c) yields

$$(1-\lambda)^2 \simeq \left(\frac{\lambda_1{}^2}{\mu_2}\right)(\mu). \tag{4.8}$$

Thus, the greater this coefficient, the larger is the decrease in the instability loads for small imperfections. This coefficient is plotted in Fig. 1 against f_1, the component of the imperfection in the mode φ_1, which is the symmetric mode [1]. In [1] it was concluded that the imperfection in the asymmetric mode ($f_1 = 0$; $f_2 = 1$) is more catastrophic than the one in the symmetric mode. From Fig. 1 it can be seen that most catastrophic imperfection is a combination of the two, and in that case the imperfection sensitivity coefficient is about one and one-quarter times larger than in the case of pure asymmetric bifurcation.

The equations corresponding to eqns (4.1) and (4.4) for three mode solutions are, for

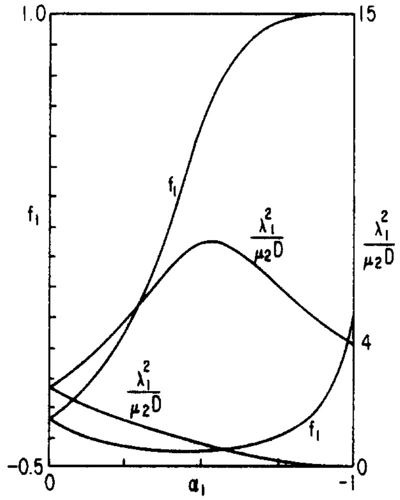

Fig. 1. Two mode interaction in imperfection sensitivity of externally pressurized spherical shell.

equilibrium [1]

$$\begin{aligned} u_1(1-\lambda)-Du_2u_3&=\mu f_1\\ u_2(1-\lambda)-Du_3u_1&=\mu f_2\\ u_3(1-\lambda)-Du_1u_2&=\mu f_3 \end{aligned} \tag{4.9a–c}$$

and for transition to instability

$$\begin{aligned} \psi_1(1-\lambda)-D(u_2\psi_3+\psi_2u_3)&=0\\ \psi_2(1-\lambda)-D(u_3\psi_1+\psi_3u_1)&=0\\ \psi_3(1-\lambda)-D(u_1\psi_2+\psi_1u_2)&=0\\ \psi_1^2+\psi_2^2+\psi_3^2&\neq 0. \end{aligned} \tag{4.10a–d}$$

Here, of course, the subscripts in u and ψ refer to bifurcation modes which are different from the ones for the two mode solution. The lowest order perturbation equations are

$$-\lambda_1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}-\frac{D}{2}\begin{bmatrix}0&\alpha_3&\alpha_2\\\alpha_3&0&\alpha_1\\\alpha_2&\alpha_1&0\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}=\mu_2\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix} \tag{4.11}$$

$$\left\{\lambda_1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+D\begin{bmatrix}0&\alpha_3&\alpha_2\\\alpha_3&0&\alpha_1\\\alpha_2&\alpha_1&0\end{bmatrix}\right\}\begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}=0 \tag{4.12}$$

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=\xi_1^2+\xi_2^2+\xi_3^2=f_1^2+f_2^2+f_3^2=1. \tag{4.13}$$

These equations have a two-parameter family of solutions with α_1 in $[-1, 1]$ and α_2 in $[-\sqrt{(1-\alpha_1^2)}, +\sqrt{(1-\alpha_1^2)}]$. For a given $(\alpha_1, \alpha_2, \alpha_3)$, eqn (4.12) is an eigenvalue problem leading to the cubic equation

$$\lambda_1^3-\lambda_1D^2+2D^3\alpha_1\alpha_2\alpha_3=0. \tag{4.14}$$

Thus the calculation proceeds in the manner already described: for a given $(\alpha_1, \alpha_2, \alpha_3)$ the values of λ_1 are obtained from eqn (4.14) and for each of these μ_2 and (f_1, f_2, f_3) are obtained from eqn (4.11). The results of these calculations are presented in Fig. 2 where λ_1^2/μ_2D is

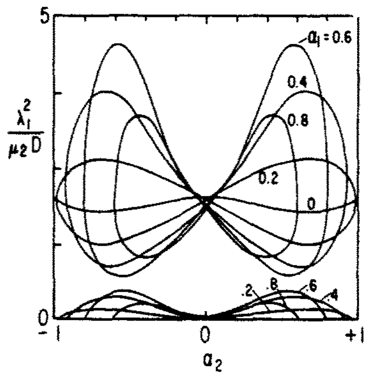

Fig. 2. Three mode interaction in imperfection sensitivity of externally pressurized spherical shell.

plotted against α_2 with α_1 as a parameter. The solutions are symmetric about the lines $\alpha_2 = 0$ and $\alpha_1 = 0$. Of course only the highest branch of these curves is important for a given α_2, since it corresponds to the maximum decrease in the buckling load. From these curves it can also be concluded that the maximum imperfection sensitivity occurs for

$$|\alpha_1| = |\alpha_2| = |\alpha_3| = \frac{1}{\sqrt{(3)}} \tag{4.15}$$

although this maximum $(\lambda_1^2/\mu_2 D \simeq 4.5)$ is less than the maximum in the two mode case $(\lambda_1^2/\mu_2 D \simeq 7.5)$. However, it is of some interest to note that the condition (4.15) leads to a polygonal deflection pattern which has sometimes been experimentally observed.

5. CONCLUSION

The theory developed in the preceding sections is an application and extension of the basic ideas of Keller[5, 6]. It is therefore possible to make it more rigorous in an appropriate function space setting although no such attempt has been made here. From application point of view this development is quite adequate and can be used for other problems in which the bifurcation point is degenerate.

It is of some interest to note that the work presented here falls under the category of what has now come to be known as catastrophe theory. Relationship between this general theory of catastrophic phenomena and structural stability has been studied recently by Thompson and Hunt[7]. However, in contradistinction to the qualitative nature of the study of the two mode interaction in[7], the method given here can be used, in principle, for obtaining quantitative results for more general cases.

REFERENCES

1. J. W. Hutchinson, Imperfection sensitivity of externally pressurized spherical shells. *J. Appl. Mech.* **34**, 49–55 (1967).
2. W. T. Koiter, Stability of elastic equilibrium. Thesis, Delft, 1945 (AFFDL-TR-70-25) (1970).
3. J. M. T. Thompson and G. W. Hunt, A theory for the numerical analysis of compound branching. *Zeitschrift für angewandte Mathematic und Physic* **22**, 1001–1015 (1971).
4. E. Reissner, A note on postbuckling behavior of pressurized shallow spherical shells, *J. Appl. Mech.* **37**, 533–534 (1970).
5. H. B. Keller and W. F. Langford, Iterations, perturbations and multiplicities for nonlinear bifurcation problems. *Arch. for Rational Mech. and Anal.* **48**, 83–108 (1972).
6. J. P. Keener and H. B. Keller, Perturbed bifurcation theory. *Arch. for Rational Mech. and Anal.* **50**, 159–175 (1973).
7. J. M. T. Thompson and G. W. Hunt, Towards a unified bifurcation theory. *J. Appl. Math. Phys.* (*ZAMP*) **26**, 581–604 (1975).
8. B. Budiansky, Theory of buckling and postbuckling of elastic structures. *Advances in Appl. Mech.* (Edited by C. S. Yih), Vol. 14. Academic Press, New York (1974).

APPENDIX

The choice of the leading term in the expansion for μ is indicated by the following considerations. Assume

$$\mu = \epsilon\mu_1 + O(\epsilon^2) \tag{A1}$$

and same expansion as in eqn (3.3) for other quantities. If this expansion is substituted in eqn (3.1) and the coefficient of ϵ is set to zero, the following equation is obtained:

$$P_{11}(v_1, \delta u) + \mu_1 Q_1(\delta u) = 0. \tag{A2}$$

The solvability condition for eqn (A2) is

$$\mu_1 Q_1(\varphi_i) = 0 \qquad i = 1, \ldots r. \tag{A3}$$

Thus μ_1 is nonzero only when the imperfection is orthogonal to *all* the buckling modes. Even if this condition is satisfied, it leads to the result

$$(\lambda - \lambda_0) = O(\mu) \tag{A4}$$

so that the structure is not highly imperfection sensitive in that small imperfections do not lead to large decrease in the buckling load. Thus for imperfection sensitivity calculations more interesting case occurs when the imperfection is not orthogonal to all the buckling modes. In such a case μ_1 is zero and the leading term in the expansion of μ is $O(\epsilon^2)$ as assumed in eqn (3.3).